

# Non-singular circulant graphs and digraphs

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**Abstract** We give necessary and sufficient conditions for a few classes of known circulant graphs and/or digraphs to be singular. The above graph classes are generalized to  $(r, s, t)$ -digraphs for non-negative integers  $r, s$  and  $t$ , and the digraph  $C_n^{i,j,k,l}$ , with certain restrictions. We also obtain a necessary and sufficient condition for the digraphs  $C_n^{i,j,k,l}$  to be singular. Some necessary conditions are given under which the  $(r, s, t)$ -digraphs are singular.

**Keywords:** Graphs, Digraphs, Circulant matrices, Primitive roots.

## 1 Introduction and preliminaries

Let  $\mathbb{Q}$  denote the set of rational numbers. Then the set of all  $n \times n$  matrices with entries from  $\mathbb{Q}$  is denoted by  $M_n(\mathbb{Q})$ . A matrix  $A \in M_n(\mathbb{Q})$  is said to be symmetric if  $A = A^t$ , where  $A^t$  denotes the transpose of the matrix  $A$  and is said to be circulant if  $a_{ij} = a_{1, j-i+1}$ , whenever  $2 \leq i \leq n$  and  $1 \leq j \leq n$ , where the subscripts are read modulo  $n$ . From the definition, it is clear that if  $A$  is circulant then for each  $i \geq 2$  the elements of the  $i$ -th row are obtained by cyclically shifting the elements of the  $(i-1)$ -th row one position to the right. So it is sufficient to specify its first row. For example, the identity matrix, denoted  $I$ , and the matrix of all 1's, denoted  $\mathbf{J}$ , are circulant matrices. Let  $W_n$  be a circulant matrix of order  $n$  with  $[0, 1, 0, \dots, 0]$  as its first row. Then the following result of Davis [4] establishes that every circulant matrix of order  $n$  is a polynomial in  $W_n$ .

**Lemma 1.1.** [4] *Let  $A \in M_n(\mathbb{Q})$ . Then  $A$  is circulant if and only if it is a polynomial over  $\mathbb{Q}$  in  $W_n$ .*

Let  $A \in M_n(\mathbb{Q})$  be a circulant matrix. Then Lemma 1.1 ensures the existence of a polynomial  $\gamma_A(x) \in \mathbb{Q}[x]$  such that  $A = \gamma_A(W_n)$ . We call  $\gamma_A(x)$ , the representer polynomial of  $A$ . For a fixed positive integer  $n$ , let  $\zeta_n$  denote a primitive  $n$ -th root of unity. That is,  $\zeta_n^n = 1$  and  $\zeta_n^k \neq 1$  for  $k = 1, 2, \dots, n-1$ . Then the following result about circulant matrices is well known.

**Lemma 1.2.** *Let  $A \in M_n(\mathbb{Q})$  be a circulant matrix with  $[a_0, a_1, \dots, a_{n-1}]$  as its first row. Then*

1.  $\gamma_A(x) = a_0 + a_1x + \dots + a_{n-1}x^{n-1} \in \mathbb{Q}[x]$ .
2. *the eigenvalues of  $A$  are given by  $\gamma_A(\zeta_n^k)$ , for  $k = 0, 1, \dots, n-1$ .*

For definitions and results related to linear algebra, algebra and/or graph theory that have been used in this paper but not have been cleared defined or stated, the readers are advised to see any standard textbook on abstract algebra and/or graph theory (for example, see [6] and/or [3]).

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Recall that a *directed graph* (in short, digraph) is an ordered pair  $X = (V, E)$  that consists of two sets  $V$ , the vertex set, and  $E$ , the edge set, where  $V$  is a non-empty set and  $E \subset V \times V$ . If  $e = (u, v) \in E$  with  $u \neq v$  then the edge  $e$  is said to be *incident* from  $u$  to  $v$ . A digraph is called a *graph* if  $(u, v) \in E$  whenever  $(v, u) \in E$ , for any two elements  $u, v \in V$ . An edge between  $u$  and  $v$  in the graph  $X$  is denoted by  $\{u, v\}$ . A graph/digraph is said to be finite, if  $|V|$  (called the *order* of  $X$ ) and  $|E|$  (called the *size* of  $X$ ) are finite. All the graphs/digraphs in this paper are finite. The adjacency matrix of a graph/digraph  $X = (V, E)$  is a  $|V| \times |V|$  matrix, denoted  $A(X) = [a_{uv}]$ , with  $a_{uv} = 1$  if  $(u, v) \in E$  and 0, otherwise. Observe that, whenever  $X$  is a graph the matrix  $A(X)$  is symmetric. For example, if  $A$  denotes the adjacency matrix of the cycle graph  $C_n$  on  $n$  vertices, then  $A$  is a circulant matrix and  $\gamma_A(x) = x + x^{n-1}$  is its representer polynomial. Therefore, for  $r = 0, 1, \dots, n-1$ , the eigenvalues of  $C_n$  are given by  $\lambda_r = 2 \cos(\frac{2\pi r}{n})$ . Throughout this paper, we assume that the greatest common divisor, in short gcd, of all the non-zero coefficients of  $\gamma_A(x)$  is 1. It is well known that  $x^n - 1 = \prod_{d|n} \Phi_d(x)$  (here  $a \mid b$  means  $a$

‘divides’  $b$ ), where  $\Phi_d(x) = \prod_{\substack{\gcd(k,d)=1 \\ 1 \leq k \leq d}} (x - \zeta_d^k) \in \mathbb{Z}[x]$  is called the  $d$ -th cyclotomic polynomial.

The polynomial  $\Phi_n(x)$ , for each positive integer  $n$ , is a monic irreducible polynomial over  $\mathbb{Q}$  and hence the minimal polynomial of  $\zeta_n$ . Also,  $\deg(\Phi_n(x)) = \varphi(n)$ , the well known Euler-totient function. Therefore, using the property of minimal polynomials, it follows that if  $f(\zeta_n) = 0$  for some  $f(x) \in \mathbb{Z}[x]$  then  $\Phi_n(x)$  divides  $f(x) \in \mathbb{Z}[x]$ . Or equivalently,  $f(\zeta_n) = 0$  for some  $f(x) \in \mathbb{Z}[x]$  if and only if there exists a polynomial  $g(x) \in \mathbb{Z}[x]$  such that  $f(x) = \Phi_n(x)g(x)$ . The next result appears on page 93 in [9].

**Lemma 1.3.** [9] *Let  $p$  be a prime number and let  $n$  be a positive integer. Then*

$$\Phi_{pn}(x) = \begin{cases} \Phi_n(x^p), & \text{if } p \mid n, \\ \frac{\Phi_n(x^p)}{\Phi_n(x)}, & \text{if } p \nmid n. \end{cases}$$

*In particular,  $\Phi_{p^k}(x) = 1 + x^{p^{k-1}} + x^{2p^{k-1}} + \dots + x^{(p-1)p^{k-1}}$  for every positive integer  $k$ .*

The following result is an application of Lemma 1.2. This result also appears in the work of Geller, Kra, Popescu & Simanca [7].

**Lemma 1.4** (Geller, Kra, Popescu & Simanca [7]). *Let  $A \in \mathbb{M}_n(\mathbb{Q})$  be a circulant matrix with  $\gamma_A(x)$  as its representer polynomial. Then the following statements are equivalent:*

1. *The matrix  $A$  is singular.*
2.  $\deg(\gcd(\gamma_A(x), x^n - 1)) \geq 1$ .

Fix a positive integer  $n$ , two distinct integers  $a$  and  $b$  and let  $s$  and  $t$  be positive integers with  $s + t = n$ . Suppose  $[\underbrace{a, a, \dots, a}_{s \text{ times}}, \underbrace{b, b, \dots, b}_{t \text{ times}}]$  is the first row of a circulant matrix  $A \in \mathbb{M}_n(\mathbb{Z})$ .

Then as a direct corollary of Lemma 1.4, one has the following result.

**Corollary 1.5** (Davis [4]). *Let  $[\underbrace{a, a, \dots, a}_{s \text{ times}}, \underbrace{b, b, \dots, b}_{t \text{ times}}]$  be the first row of the circulant matrix*

$A \in \mathbb{M}_n(\mathbb{Q})$ . *Then*

$$\det(A) = \begin{cases} (sa + tb)(a - b)^{n-1}, & \text{if } \gcd(s, n) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

We now state a couple of known results that directly follow from Corollary 1.5.

**Lemma 1.6.** *The complete graph  $K_n$ , for  $n \geq 2$ , is non-singular.*

*Proof.* Let  $A$  be the adjacency matrix of complete graph  $K_n$ . Then  $[0, 1, 1, \dots, 1]$  is the first row of  $A$ . Hence the result follows from Corollary 1.5.  $\square$

As a second application, we consider a particular class of circulant matrices that appeared in the work of Searle [11]. He considered the circulant matrices that have  $[h_0, h_1, \dots, h_{k-1}, \underbrace{0, \dots, 0}_{n-k}]$  as its first row, where  $h_0 \neq 0$  and  $h_{k-1} \neq 0$ . The above class of matrices was called a *k-element circulant matrix*. Since we are looking at digraphs, we assume  $h_0 = 1 = h_{k-1}$ . With an abuse of notation, the circulant matrix with  $[\underbrace{1, 1, \dots, 1}_k, \underbrace{0, 0, \dots, 0}_{n-k}]$  as its first row will be called a *k-element circulant digraph*. With this notation, the second application of Corollary 1.5 is stated below.

**Lemma 1.7.** *Let  $X$  be a k-element circulant digraph on  $n$  vertices. Then  $X$  is non-singular if and only if  $\gcd(n, k) = 1$ .*

We now rephrase Lemma 1.4 in terms of cyclotomic polynomials. Let  $A \in \mathbb{M}_n(\mathbb{Z})$  be a circulant matrix with  $[a_0, a_1, \dots, a_{n-1}]$  as its first row. Then  $\gamma_A(x) = a_0 + a_1x + \dots + a_{n-1}x^{n-1}$  is the representer polynomial of  $A$ . Now, suppose that  $a_0 = 0$  and let  $k$  be the smallest positive integer such that  $a_k \neq 0$ . Then  $\gamma_A(x) = x^k \Gamma_A(x)$ , for some polynomial  $\Gamma_A(x) \in \mathbb{Z}[x]$ . In this case, it follows that the matrix  $A$  is non-singular if and only if  $\gcd(\Gamma_A(x), x^n - 1) = 1$  as  $\gcd(x^k, x^n - 1) = 1$ . This observation leads to the next remark.

**Remark 1.8.** *Let  $A \in M_n(\mathbb{Z})$  be a circulant matrix and for each fixed positive integer  $k$  consider the matrix  $W_n^k A$ . Then  $A$  is singular (non-singular) if and only if  $W_n^k A$  is singular (non-singular). That is, if we want to study singularity/non-singularity of a matrix  $A$  then it is enough to study  $\Gamma_A(x)$ .*

Using Remark 1.8 and Lemma 1.4, the following result is immediate and hence the proof is omitted.

**Lemma 1.9.** *Let  $A$  be a circulant digraph of order  $n$  and let  $\Gamma_A(x)$  be the polynomial defined above. Then  $A$  is singular if and only if  $\Phi_d(x) \mid \Gamma_A(x)$ , for some divisor  $d \neq 1$  of  $n$ .*

As an immediate corollary of Lemma 1.9, we have the following result.

**Corollary 1.10.** *Let  $p$  be a prime and let  $k$  be a positive integer with  $p \nmid k$ . Also, let  $X$  be a k-regular circulant graph/digraph on  $p^\ell$  vertices, for some positive integer  $\ell$ . Then  $X$  is non-singular.*

*Proof.* Using Lemma 1.9, we just need to show that  $\Phi_d(x) \nmid \Gamma_A(x)$  for every  $d \mid p^\ell, d \neq 1$ . Let if possible,  $\Gamma_A(x) = \Phi_d(x)g(x)$  for some  $g(x) \in \mathbb{Z}[x]$ . Using Lemma 1.3, we have  $\Phi_d(1) = p$  for every  $d \mid p^\ell, d \neq 1$ . As  $g(x) \in \mathbb{Z}[x]$ ,  $g(1) \in \mathbb{Z}$ . Thus, we get

$$k = \Gamma_A(1) = \Phi_d(1)g(1) = p g(1).$$

A contradiction to our assumption that  $p \nmid k$ . Thus the proof of the result is complete.  $\square$

The remaining part of this paper consists of two more sections that are mainly concerned with applications of Lemma 1.9. Section 2 gives necessary and sufficient conditions for a few classes of circulant graphs to be non-singular and Section 3 gives possible generalization of the results studied in Section 2.

Before proceeding to Section 2, recall that for a graph  $X = (V, E_1)$ , the complement graph of  $X$ , denoted  $X^c = (V, E_2)$ , is a graph in which  $(u, v) \in E_2$  whenever  $(u, v) \notin E_1$  and vice versa, for every  $u \neq v \in V$ . Note that  $(u, u)$  is neither an element of  $E_1$  nor an element of  $E_2$ . Also, a graph  $X$  is circulant if and only if  $X^c$  is circulant and if  $A$  is the adjacency matrix of  $X$  then the adjacency matrix of  $X^c$  is given by  $\mathbf{J} - A - I$ .

## 2 Some Singular Circulant Graphs

This section is devoted to finding necessary and sufficient conditions for a few classes of circulant graphs to be singular or not. Before proceeding to these results, we show that the adjacency matrix  $A$  of a circulant graph on  $n$  vertices is a polynomial in  $W_n + W_n^{-1}$ , the adjacency matrix of  $C_n$ , the cycle graph on  $n$  vertices. To do so, we need the following definition.

**Definition 2.11.** Let  $v_1, v_2, \dots, v_n$  be the vertices of a connected graph  $X$ . If  $d$  is the diameter of  $X$  then, for  $0 \leq k \leq d$ , the  $k$ -th distance matrix of  $X$ , denoted  $A_k(X)$ , is defined as

$$(A_k(X))_{rs} = \begin{cases} 1, & \text{if } d(v_r, v_s) = k, \\ 0, & \text{otherwise,} \end{cases}$$

where  $d(u, v)$  is the distance between the vertices  $u, v \in V$ .

For example,  $\tau = \lfloor \frac{n}{2} \rfloor$  and consider the cycle graph  $C_n$ . Also, let us write  $A_k$  to denote the distance matrices  $A_k(C_n)$ , for  $0 \leq k \leq \tau$ . Then, for  $1 \leq i < \tau$ ,  $A_i = W_n^i + W_n^{n-i}$  and

$$A_\tau = \begin{cases} W_n^\tau, & \text{if } n \text{ is even,} \\ W_n^\tau + W_n^{n-\tau}, & \text{if } n \text{ is odd.} \end{cases} \quad (2.1)$$

The identity

$$(x^k + x^{-k}) = (x + x^{-1})(x^{k-1} + x^{1-k}) - (x^{k-2} + x^{2-k})$$

enables us to readily establish, by mathematical induction, that  $x^k + x^{-k}$  is a monic polynomial in  $x + x^{-1}$  of degree  $k$  with integral coefficients. Also, for  $n$  even,  $2\tau = n$  and hence  $W_n^\tau = \frac{W_n^\tau + W_n^{-\tau}}{2}$ . Consequently,  $A_i$ 's, for  $1 \leq i \leq \tau$ , are polynomials of degree  $\leq i$ , in  $A$ , the adjacency matrix of  $C_n$ , over  $\mathbb{Q}$ . Now, let  $B$  be a symmetric circulant matrix with representer polynomial  $\gamma_B(x) = \sum_{i=0}^{n-1} b_i x^i$ . Then by definition,  $B = \gamma_B(W_n) = \sum_{i=0}^{n-1} b_i W_n^i$  and  $B^t = \sum_{i=0}^{n-1} b_i W_n^{n-i}$ . But  $B$  is

symmetric implies that  $B = B^t$  and therefore,  $b_i = b_{n-i}$ , for  $1 \leq i \leq n-1$ . Thus,  $B = \sum_{i=0}^{\tau} b_i A_i$  and hence we see that the adjacency matrix of any circulant graph is a polynomial in  $A$ , the adjacency matrix of  $C_n$ , over  $\mathbb{Q}$ .

For  $1 \leq i \leq \tau$ , let us denote the graph with adjacency matrix  $A_i$  as  $C_n^i$ . Then observe that  $C_n^1 = C_n$  is the cycle graph on  $n$  vertices. Also, note that the corresponding representer polynomials, for  $1 \leq i < \tau$ , are given by  $\gamma_{A_i}(x) = x^i (1 + x^{n-2i})$  and  $\gamma_{A_\tau}(x) = x^\tau$ , if  $n$  is even, and  $\gamma_{A_\tau}(x) = x^\tau (1 + x)$ , if  $n$  is odd. The next result uses the above notations and observations to give a necessary and sufficient condition for the graphs  $C_n^i$ , for  $1 \leq i \leq \tau$ , to be singular.

**Lemma 2.12.** Fix a positive integer  $n \geq 3$  and let  $1 \leq i \leq \tau = \lfloor \frac{n}{2} \rfloor$ . Then the graph  $C_n^i$  is singular if and only if  $n$  is a multiple of 4 and  $\gcd(i, \frac{n}{2}) \mid \frac{n}{4}$ .

*Proof.* Using the discussion above,  $\Gamma_{A_i}(x) = 1 + x^{n-2i}$ , for  $1 \leq i < \tau$  and  $\Gamma_{A_\tau}(x) = 1 + x$ , if  $n$  is odd, and  $\Gamma_{A_\tau}(x) = 1$ , if  $n$  is even. If  $n$  is odd then  $\zeta_n^k \neq -1$  for any  $k, 1 \leq k \leq n-1$ . Hence,  $A_\tau$  is non-singular for all  $n$ . So, we need to consider  $\Gamma_{A_i}(x) = 1 + x^{n-2i}$ , for  $1 \leq i < \tau$ .

In this case,  $A_i$  is singular if and only if  $\Gamma_{A_i}(\zeta_n^k) = 0$ , for some  $k, 1 \leq k \leq n-1$ . That is, we need  $(\zeta_n^k)^{n-2i} = -1$ . Or equivalently, we need  $2ki \equiv \frac{n}{2} \pmod{n}$ . That is,  $ki \equiv \frac{n}{4} \pmod{\frac{n}{2}}$ . Therefore, it follows that 4 divides  $n$  and  $\gcd(i, \frac{n}{2}) \mid \frac{n}{4}$ .  $\square$

**Remark 2.13.** We can rewrite the condition in Lemma 2.12 as follows:

The graph  $C_n^i$  is singular if and only if the following conditions are satisfied:

1.  $n$  is a multiple of 4, and
2. if  $s$  is the largest positive integer such that  $2^s$  divides  $n$  then  $i$  is an odd multiple of  $2^t$  for some  $t, 0 \leq t \leq s-2$ .

As an immediate consequence of Lemma 2.12, we have the following corollary.

**Corollary 2.14.** Let  $C_n$  be the cycle graph on  $n$  vertices. Then  $C_n$  is singular if and only if  $4 \mid n$ .

The next result gives a necessary and sufficient condition for the complement graph  $(C_n^i)^c$  of  $C_n^i$  to be singular.

**Lemma 2.15.** Fix a positive integer  $n \geq 4$ . Then the graph

1.  $(C_n^\tau)^c$  is singular if and only if  $n$  is even or  $n \equiv 3 \pmod{6}$ .
2.  $(C_n^i)^c$ , for  $1 \leq i < \tau$ , is singular if and only if  $3 \mid n$  and  $\gcd(i, n) \mid \frac{n}{3}$ .

*Proof.* Let  $n$  be even. Then, using definition of complement of a graph, the adjacency matrix of  $(C_n^\tau)^c$ , say  $A$ , is given by  $\mathbf{J} - A_\tau - I$ . Hence,  $\gamma_A(x) = 1 + x + \cdots + x^{n-1} - 1 - \gamma_{A_\tau}(x)$  is the representer polynomial of  $(C_n^\tau)^c$ . Thus,

$$\gamma_A(x) = \frac{x^n - 1}{x - 1} - (1 + x^\tau)$$

and hence  $\gamma_A(\zeta_n) = 0$  ( $n$  is even). Thus, the graph  $(C_n^\tau)^c$  is singular, whenever  $n$  is even. For  $n$  odd, it can be checked that

$$\gamma_A(x) = \frac{x^n - 1}{x - 1} - (1 + x^\tau + x^{\tau+1}).$$

Consequently,  $(C_n^\tau)^c$  is singular if and only if  $\gamma_A(\zeta_n^k) = 0$ , for some  $k, 1 \leq k \leq n-1$ . Or equivalently,  $1 + (\zeta_n^k)^\tau + (\zeta_n^k)^{\tau+1} = 0$ , for some  $k, 1 \leq k \leq n-1$ . This is equivalent to the statement that  $\zeta_n^{k\tau}$  is a primitive 3-rd root of unity. Thus,  $k\tau \equiv \frac{n}{3} \pmod{n}$ , or equivalently,  $\gcd(\tau, n) \mid \frac{n}{3}$ . Thus,  $n \equiv 3 \pmod{6}$  and in this case,  $\gcd(\tau, n)$  indeed divides  $\frac{n}{3}$ .

Now assume that  $1 \leq i < \tau$ . In this case, if  $A$  is the adjacency matrix of  $(C_n^i)^c$ , then  $A = \mathbf{J} - A_i - I$ . Consequently, its representer polynomial is  $\gamma_A(x) = \frac{x^n - 1}{x - 1} - (1 + x^i + x^{n-i})$ . Thus,  $(C_n^i)^c$  is singular if and only if  $\gamma_A(\zeta_n^k) = 0$ , for some  $k, 1 \leq k \leq n-1$ . Or equivalently,  $1 + \zeta_n^{ki} + \zeta_n^{-ki} = 0$ , for some  $k, 1 \leq k \leq n-1$ . That is,  $\zeta_n^{ki}$  is a primitive 3-rd root of unity. Thus, using the argument similar to one in the first part, one has  $(C_n^i)^c$  is singular if and only if  $3 \mid n$  and  $\gcd(i, n) \mid \frac{n}{3}$ .  $\square$

As an immediate consequence of Lemma 2.15, we have the following corollary.

**Corollary 2.16.** *Fix a positive integer  $n$  and let  $C_n^c$  be the complement graph of the cycle graph  $C_n$ . Then the complement graph  $C_n^c$  is singular if and only if  $3 \mid n$ .*

We now obtain necessary and sufficient conditions for non-singularity of circulant graphs that were studied by Ruivivar [10]. In [10], the author studied two classes of graphs. For the sake of notational clarity, his notations have been slightly modified. Fix a positive integer  $n \geq 3$  and let  $1 \leq r < \tau = \lfloor \frac{n}{2} \rfloor$ . The first class of circulant graphs, denoted  $C_n^{(r)}$ , has the same vertex set as the vertex set of the cycle  $C_n$  and  $\{x, y\}$  is an edge whenever the length of the smallest path from  $x$  to  $y$  in  $C_n$  is at most  $r$ . He called these graphs the  $r$ -th power graph of the cycle graph  $C_n$ . Note that  $C_n^{(\tau)}$  is the complete graph. The second class of graphs, denoted  $C(2n, r)$  is a graph on  $2n$  vertices and its adjacency matrix is the sum of the adjacency matrices of  $C_{2n}^{(r)}$  and  $C_{2n}^n$ , where  $1 \leq r < n$ . The next result appears as Theorem 2.2 of [10]. We give a separate proof for the sake of completeness.

**Theorem 2.17** (Ruivivar [10]). *Let  $n \geq 3$  and let  $1 \leq r < \lfloor \frac{n}{2} \rfloor$ . Then the graph  $C_n^{(r)}$  is singular if and only if one of the following conditions hold:*

1.  $\gcd(n, r) > 1$
2.  $\gcd(n, r) = 1$ ,  $n$  is even and  $\gcd(r+1, n)$  divides  $\frac{n}{2}$ .

*Proof.* Let  $A$  be the adjacency matrix of the graph  $C_n^{(r)}$ . Then, by definition, the first row of  $A$  equals  $[0, \underbrace{1, 1, \dots, 1}_r, \underbrace{0, 0, \dots, 0}_{n-2r-1}, \underbrace{1, 1, \dots, 1}_r]$  and  $\gamma_A(x) = x\Gamma_A(x)$ , where

$$\Gamma_A(x) = [1 + x + \dots + x^{r-1}] + x^{n-r-1}[1 + x + \dots + x^{r-1}] = \frac{x^r - 1}{x - 1}(1 + x^{n-r-1}).$$

Therefore,  $C_n^{(r)}$  is singular if and only if  $\Gamma_A(\zeta_n^d) = 0$ , for some  $d, 1 \leq d \leq n-1$ . Or equivalently either  $(\zeta_n^d)^r - 1 = 0$  or  $1 + (\zeta_n^d)^{n-r-1} = 0$ .

If  $(\zeta_n^d)^r - 1 = 0$  then  $\gcd(r, n) > 1$  is the required condition as  $1 \leq d \leq n-1$ . If  $\gcd(r, n) = 1$  then we need  $1 + (\zeta_n^d)^{n-r-1} = 0$ . This implies that  $d(r+1) \equiv \frac{n}{2} \pmod{n}$ . Which in turn gives the required result.

Thus, the proof of the theorem is complete.  $\square$

The following result can be seen as a corollary to Lemma 1.7. But an idea of the proof is given for completeness.

**Corollary 2.18.** *Let  $n \geq 3$  and let  $1 \leq r < \lfloor \frac{n}{2} \rfloor$ . Then the graph  $(C_n^{(r)})^c$  is non-singular if and only if  $\gcd(n, 2r+1) = 1$ .*

*Proof.* Let  $A$  be the adjacency matrix of  $(C_n^{(r)})^c$ . Then  $[0, 0, \dots, 0, \underbrace{1, \dots, 1}_{r+1}, \underbrace{0, 0, \dots, 0}_{n-2r-1}, \underbrace{0, 0, \dots, 0}_r]$  is the first row of  $A$ . Thus, using Remark 1.8,  $A$  is singular if and only if the circulant matrix with  $[1, 1, \dots, 1, \underbrace{0, 0, \dots, 0}_{n-2r-1}, \underbrace{0, 0, \dots, 0}_{2r+1}]$  as its first row is singular. Thus, using Lemma 1.7,  $A$  is singular if and only if  $\gcd(2r+1, n) > 1$ . Hence, the required result follows.  $\square$

Before proceeding with the next result that gives a necessary and sufficient condition for the graph  $C(2n, r)$  to be singular, we state a result that appears as Proposition 1 in Kurshan & Odlyzko [8]

**Lemma 2.19** (Kurshan & Odlyzko [8]). *Let  $m$  and  $n$  be positive integers with  $m \neq n$  and let  $\zeta_n$  be a primitive  $n$ -root of unity. Then there exists a unit  $u \in \mathbb{Z}[\zeta_n]$  dependent on  $m, n$  and  $\zeta_n$  such that*

$$\Phi_m(\zeta_n) = \begin{cases} pu, & \text{if } \frac{m}{n} = p^\alpha, \quad p \text{ a prime}, \quad \alpha > 0; \\ (1 - \zeta_{p^\alpha})u, & \text{if } \frac{m}{n} = p^{-\alpha}, \quad p \text{ a prime}, \quad \alpha > 0; \quad p \nmid m; \\ (1 - \zeta_{p^\alpha+1})^{p-1}u, & \text{if } \frac{m}{n} = p^{-\alpha}, \quad p \text{ a prime}, \quad \alpha > 0; \quad p \mid m; \\ u, & \text{otherwise.} \end{cases}$$

**Theorem 2.20.** *Let  $n$  and  $r$  be positive integers such that the circulant graph  $C(2n, r)$  is well defined. Then the circulant graph  $C(2n, r)$  is singular if and only if  $\gcd(n, 2r+1) \geq 3$ .*

*Proof.* Let  $A$  be the adjacency matrix of the graph  $C(2n, r)$ . Then observe that the first row of  $A$  equals  $[0, \underbrace{1, 1, \dots, 1}_r, \underbrace{0, 0, \dots, 0}_{n-r-1}, 1, \underbrace{0, 0, \dots, 0}_{n-r-1}, \underbrace{1, 1, \dots, 1}_r]$ . Consequently,

$$\gamma_A(x) = x + x^2 + \dots + x^r + x^n + x^{2n-r} + \dots + x^{2n-1} = x\Gamma_A(x)$$

and

$$\begin{aligned} (x-1)\Gamma_A(x) &= x^r - 1 + x^{n-1}(x-1) + x^{2n-r-1}(x^r - 1) \\ &= x^r(1 - x^{2n-2r-1}) + (x^n - 1) - (x^{n-1} - x^{2n-1}) \\ &= (x^n - 1)(x^{n-1} + 1) - x^r(x^{2n-2r-1} - 1). \end{aligned}$$

Now, let us assume that  $\gcd(n, 2r+1) = d \geq 3$ . Then  $(\zeta_{2n}^{2n/d} - 1)\Gamma_A(\zeta_{2n}^{2n/d}) = 0$  as

$$\left(\zeta_{2n}^{2n/d}\right)^n = (\zeta_{2n}^{2n})^{n/d} = 1 = (\zeta_{2n}^{2n})^{(2r+1)/d} = \left(\zeta_{2n}^{2n/d}\right)^{2r+1} = \left(\zeta_{2n}^{2n/d}\right)^{2n-2r-1}.$$

Hence, the circulant graph  $C(2n, r)$  is singular.

Conversely, let us assume that the graph  $C(2n, r)$  is singular. This implies that there exists an eigenvalue of  $C(2n, r)$  that equals zero. That is, there exists a  $k$ ,  $1 \leq k \leq 2n-1$ , such that  $\gamma_A(\zeta_{2n}^k) = 0$ . We will now show that if  $\gcd(n, 2r+1) = 1$  then the expression  $(x-1)\Gamma_A(x)$  evaluated at  $x = \zeta_{2n}^k$  can never equal zero, for any  $k$ ,  $1 \leq k \leq 2n-1$ , and this will complete the proof of the result.

We need to consider two cases depending on whether  $k$  is odd or  $k$  is even. Let  $k$  be even, say  $k = 2m$ , for some  $m$ ,  $1 \leq m < n$ . Then evaluating  $(x-1)\Gamma_A(x)$  at  $x = \zeta_{2n}^{2m}$  and using  $\gcd(n, 2r+1) = 1$  leads to

$$\begin{aligned} [(\zeta_{2n}^{2m})^n - 1] \left[ (\zeta_{2n}^{2m})^{(n-1)} + 1 \right] - (\zeta_{2n}^{2m})^r \left[ (\zeta_{2n}^{2m})^{2n-2r-1} - 1 \right] \\ = -(\zeta_{2n}^{2m})^r \left[ (\zeta_{2n}^{2m})^{-(2r+1)} - 1 \right] \neq 0. \end{aligned}$$

Now, let  $k$  be odd, say  $k = 2m + 1$ , for some  $m$ ,  $0 \leq m \leq n - 1$ . Then evaluating  $(x - 1)\Gamma_A(x)$  at  $x = \zeta_{2n}^{2m+1}$  leads to

$$\begin{aligned}
& \left[ (\zeta_{2n}^{2m+1})^n - 1 \right] \left[ (\zeta_{2n}^{2m+1})^{(n-1)} + 1 \right] - (\zeta_{2n}^{2m+1})^r \left[ (\zeta_{2n}^{2m+1})^{2n-2r-1} - 1 \right] \\
&= -2 \left[ -\zeta_{2n}^{-(2m+1)} + 1 \right] - \zeta_{2n}^{-(2m+1)(r+1)} \left[ 1 - \zeta_{2n}^{(2m+1)(2r+1)} \right] \\
&= -\frac{\zeta_{2n}^{2m+1} - 1}{\zeta_{2n}^{(2m+1)(r+1)}} \left[ -2\zeta_{2n}^{(2m+1)r} + \frac{\zeta_{2n}^{(2m+1)(2r+1)} - 1}{\zeta_{2n}^{(2m+1)} - 1} \right] \\
&= -\frac{\zeta_{2n}^{2m+1} - 1}{\zeta_{2n}^{(2m+1)(r+1)}} \left[ -2\zeta_{2n}^{(2m+1)r} + \prod_{\ell|(2r+1), \ell \neq 1} \Phi_\ell(\zeta_{2n}^{2m+1}) \right] \tag{2.2}
\end{aligned}$$

Note that,  $\zeta_{2n}^{2m+1}$  is a  $d$ -th primitive root of unity, for some  $d$  dividing  $2n$ . As  $\gcd(2r+1, 2n) = 1$ ,  $\gcd(2r+1, d) = 1$ . Thus, using Lemma 2.19, we get  $\prod_{\ell|(2r+1), \ell \neq 1} \Phi_\ell(\zeta_{2n}^{2m+1})$  is a unit in  $\mathbb{Z}[\zeta_d]$ .

That is,  $\left| \prod_{\ell|(2r+1), \ell \neq 1} \Phi_\ell(\zeta_{2n}^{2m+1}) \right| = 1$ . Hence, in Equation (2.2), the term in the parenthesis cannot be zero. Thus, we have proved the result for the odd case as well.

Thus, the proof of the result is complete.  $\square$

**Remark 2.21.** We would like to mention here that the necessary part of Theorem 2.20 was stated and proved by Ruivivar (see Theorem 2.1 in [10]).

We will now try to understand the complement graph  $C(2n, r)^c$  of  $C(2n, r)$ .

**Lemma 2.22.** Let  $n$  and  $r$  be positive integers such that the circulant graph  $C(2n, r)$  is well defined. Then  $C(2n, r)^c$  is non-singular if and only if the following conditions hold:

1.  $n$  and  $r$  have the same parity,
2.  $\gcd(n, r+1) = 1$ , and
3. the highest power of 2 dividing  $n$  is strictly smaller than the highest power of 2 dividing  $n - r$ .

*Proof.* Let  $A$  be the adjacency matrix of  $C(2n, r)^c$ . Then  $\underbrace{[0, 0, \dots, 0]_{r+1}}_{r+1} \underbrace{[1, 1, \dots, 1]_{n-r-1}}_{n-r-1} \underbrace{[0, 1, 1, \dots, 1]_{n-r-1}}_{n-r-1} \underbrace{[0, 0, \dots, 0]_r}_r$  is the first row of  $A$ . Note that

$$\Gamma_A(x) = (1 + x^{n-r}) \frac{x^{n-r-1} - 1}{x - 1}.$$

Now, let us assume that the graph  $C(2n, r)^c$  is non-singular. This means that  $\Gamma_A(\zeta_{2n}^k) \neq 0$ , for any  $k = 1, 2, \dots, 2n - 1$ .

Note that if  $n$  and  $r$  have opposite parity then  $\gcd(2n, n - r - 1) = d \geq 2$  and hence  $\Gamma_A(\zeta_{2n}^{2n/d}) = 0$ . Also, if  $n$  and  $r$  have the same parity and  $\gcd(n, r+1) = d > 2$  then  $n - r - 1$  is odd and  $\gcd(2n, n - r - 1) = \gcd(n, n - r - 1) = \gcd(n, r+1) = d$ . Hence, in this case again,  $\Gamma_A(\zeta_{2n}^{2n/d}) = 0$ .



Now, the only case that we need to check is the following:  
 $n$  and  $r$  have the same parity,  $\gcd(n, r+1) = 1$  and the highest power of 2 dividing  $n$  is greater than or equal to the highest power of 2 dividing  $n-r$ .

As  $n$  and  $r$  have the same parity and  $\gcd(n, r+1) = 1$ , we get  $\gcd(2n, n-r-1) = 1$  and thus

$$(\zeta_{2n}^k)^{n-r-1} - 1 \neq 0, \text{ for any } k = 1, 2, \dots, 2n-1.$$

Thus, we need to check for the condition on  $k$  so that  $1 + (\zeta_{2n}^k)^{n-r} = 0$ . This is true if and only if  $\gcd(2n, n-r) \mid n$ , or equivalently, the highest power of 2 dividing  $n$  is greater than or equal to the highest power of 2 dividing  $n-r$ .

Thus, we have the required result.  $\square$

**Remark 2.23.** Observe that using Lemma 2.22, the graph  $C(2n, r)^c$  is non-singular, whenever  $n$  and  $r$  are both odd and  $\gcd(n, r+1) = 1$ . Such numbers can be easily computed. For example, a class of such graphs can be obtained by choosing two positive integers  $s$  and  $t$  with  $s > t$  and defining  $n = 2^s - 2^t + 1$  and  $r = 2^t - 1$ .

### 3 Generalizations

In this section, we look at a few classes of graphs/digraphs, which are generalizations of the graphs that appear in Section 2. We first start with a class of circulant digraphs.

Consider a circulant matrix  $A$  whose first row contains  $r$  and  $s$  consecutive 1's separated by  $t$  consecutive 0's, where each of  $r, s$  and  $t$  are non-negative integers. That is, the vector  $\underbrace{[1, 1, \dots, 1]}_r, \underbrace{[0, 0, \dots, 0]}_t, \underbrace{[1, 1, \dots, 1]}_s, \underbrace{[0, 0, \dots, 0]}_{n-(r+t+s)}$  is the first row of  $A$ . If  $s = 0$ , then it is an  $r$ -element circulant digraph studied in Lemma 1.7. These circulant digraphs will be called an  $(r, s, t)$ -element circulant digraph. The next result gives a few conditions under which the  $(r, s, t)$ -element circulant digraph is singular.

**Lemma 3.24.** Let  $X$  be an  $(r, s, t)$ -element circulant digraph on  $n$  vertices. Then the graph  $X$  is singular if

1.  $\gcd(n, s, r) > 1$ , or
2.  $\gcd(n, s) = 1$  and one of the following condition holds:
  - (a) there exists  $d \geq 2$  such that  $d \mid t$  and  $s = \ell r$ , for some positive integer  $\ell \equiv -1 \pmod{d}$ .
  - (b)  $n$  is even, there exists an even integer  $d$  such that  $(r+t)$  is an odd multiple of  $\frac{d}{2}$  and  $s = \ell r$ , for some positive integer  $\ell \equiv 1 \pmod{d}$ .

*Proof. Proof of Part 1:* Observe that the representer polynomial of the  $(r, s, t)$ -element circulant digraph is given by

$$\begin{aligned} \gamma_A(x) &= 1 + x + \dots + x^{r-1} + x^{r+t} + \dots + x^{r+s+t-1} \\ &= \frac{x^r - 1}{x - 1} + x^{r+t} \frac{x^s - 1}{x - 1}. \end{aligned}$$

Or equivalently,

$$(x-1)\gamma_A(x) = (x^r - 1) + x^{r+t}(x^s - 1). \quad (3.3)$$

Let  $\gcd(n, r, s) = k > 1$ . Then it can be easily checked that  $\zeta_n^{n/k}$  is a root of Equation (3.3). Thus,  $X$  is singular. This completes the proof of the first part.

**Proof of Part 2.2a:** Let us assume that  $\gcd(n, s) = 1$ . Also, let us assume that there exists a positive integer  $d \geq 2$  such that  $d \mid t$  and  $s = \ell r$ , for some positive integer  $\ell \equiv -1 \pmod{d}$ . So, there exists  $\beta \in \mathbb{Z}$  such that  $\ell = \beta d - 1$ . In this case, using Equation (3.3), we get

$$\begin{aligned} (\zeta_n^{(n/d)} - 1)\gamma_A(\zeta_n^{(n/d)}) &= (\zeta_n^{(rn/d)} - 1) \left( 1 + \zeta_n^{(r+t)n/d} \frac{\zeta_n^{\ell(rn/d)} - 1}{\zeta_n^{(rn/d)} - 1} \right) \\ &= (\zeta_n^{rn/d} - 1) \left( 1 + \zeta_n^{(r+t)n/d} \frac{\zeta_n^{-(rn/d)} - 1}{\zeta_n^{(rn/d)} - 1} \right) \\ &= (\zeta_n^{rn/d} - 1) (1 - \zeta_n^{(tn/d)}). \end{aligned}$$

As  $d \mid t$ ,  $\gamma_A(\zeta_n^{n/d}) = 0$ . That is, we get the required result in this case as well.

**Proof of Part 2.2b:** Let us assume that  $\gcd(n, s) = 1$ ,  $n = 2m$ . Also, let us assume that there exists an even positive integer  $d$  such that  $r + t$  is an odd multiple of  $\frac{d}{2}$  and  $s = \ell r$ , for some positive integer  $\ell \equiv 1 \pmod{d}$ . Then there exists  $\beta \in \mathbb{Z}$  such that  $\ell = \beta d + 1$ . In this case, using Equation (3.3), we get

$$\begin{aligned} (\zeta_n^{(n/d)} - 1)\gamma_A(\zeta_n^{(n/d)}) &= (\zeta_n^{(rn/d)} - 1) \left( 1 + \zeta_n^{(r+t)n/d} \frac{\zeta_n^{\ell(rn/d)} - 1}{\zeta_n^{(rn/d)} - 1} \right) \\ &= (\zeta_n^{rn/d} - 1) \left( 1 + \zeta_n^{(r+t)n/d} \frac{\zeta_n^{(rn/d)} - 1}{\zeta_n^{(rn/d)} - 1} \right) \\ &= (\zeta_n^{rn/d} - 1) (1 + \zeta_n^{(r+t)n/d}). \end{aligned}$$

Thus, under the given conditions, the corresponding digraph  $X$  is singular.

Hence, the proof of the lemma is complete.  $\square$

Thus, the above result gives conditions under which the generalized  $(r, s, t)$ -digraphs, for non-negative values of  $r, s$  and  $t$ , are singular. We will now define another class of circulant digraphs and obtain conditions under which the circulant digraphs are singular. These graphs are also a generalization of the graphs studied in Lemma 1.7.

Let  $i, j, k$  and  $\ell$  be non-negative integers such that  $j > \ell$  and  $kj + i + \ell < n$ . Consider a class of circulant digraphs, denoted  $C_n^{i,j,k,\ell}$ , that has  $\gamma_{A(C_n^{i,j,k,\ell})}(x) = \sum_{t=0}^k \sum_{s=i}^{i+\ell} x^{s+tj}$  as its representer polynomial. Then

$$\begin{aligned} \gamma_{A(C_n^{i,j,k,\ell})}(x) &= x^i (1 + x + \cdots + x^\ell) (1 + x^j + x^{2j} + \cdots + x^{kj}) \\ &= x^i \frac{x^{\ell+1} - 1}{x - 1} \cdot \frac{x^{(k+1)j} - 1}{x^j - 1} \\ &= x^i \prod_{s|\ell+1, s \neq 1} \Phi_s(x) \cdot \prod_{t|(k+1)j, t \nmid j} \Phi_t(x). \end{aligned} \tag{3.4}$$

Hence, we have the following theorem which we state without proof.

**Theorem 3.25.** *Let  $i, j, k$  and  $\ell$  be non-negative integers with  $j > \ell$  and  $kj + i + \ell < n$ . Then the circulant digraph  $C_n^{i,j,k,\ell}$ , defined above, is singular if and only if either  $\gcd(\ell + 1, n) \geq 2$  or  $\gcd(k + 1, \frac{n}{\gcd(n,j)}) \geq 2$ .*

**Remark 3.26.** *Note that we can vary the non-negative integers  $i, j, k$  and  $\ell$  to define quite a few class of circulant digraphs. For example, it can be seen that the graphs  $G(r, t)$  that are given by Doob [5] are a particular case of the above class. Also, it can be easily verified that Theorem 3.25 is a generalization of Lemma 1.7.*

## Conclusion

In the first section, we have obtained necessary and sufficient conditions for a few known classes of circulant graphs/digraphs to be singular. We found these necessary and sufficient conditions by using Lemma 1.9. The graphs/digraphs that were studied in Section 2 have been generalized to  $(r, s, t)$ -circulant digraphs for non-negative integers  $r, s$  and  $t$ , and the circulant digraph  $C_n^{i,j,k,\ell}$ , under certain restrictions. A necessary and sufficient condition for the digraphs  $C_n^{i,j,k,\ell}$  to be singular is also obtained. Some necessary conditions are given under which the  $(r, s, t)$ -circulant digraphs are singular.

It will be nice to obtain necessary and sufficient conditions for the generalized  $(r, s, t)$ -digraphs to be singular.

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